

EQUIVARIANT VECTOR BUNDLES ON COMPLETE SYMMETRIC VARIETIES OF MINIMAL RANK

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ABSTRACT. Let X be the wonderful compactification of a complex symmetric space G/H of minimal rank. For a point $x \in G$, denote by Z be the closure of BxH/H in X , where B is a Borel subgroup of G . The universal cover of G is denoted by \tilde{G} . Given a \tilde{G} equivariant vector bundle E on X , we prove that E is nef (respectively, ample) if and only if its restriction to Z is nef (respectively, ample). Similarly, E is trivial if and only if its restriction to Z is so.

1. INTRODUCTION

Let σ be an involution of a semisimple adjoint type algebraic group G over \mathbb{C} , and let $H = G^\sigma$ be the corresponding fixed point locus. De Concini and Procesi constructed a smooth projective variety

$$X = \overline{G/H}$$

equipped with an action of G , that contains an open dense G -orbit G/H [DP]. This X is known as the wonderful compactification of the symmetric space G/H .

Richardson and Springer described the B -orbits in G/H in terms of the combinatorics of the Weyl group W , where B is a Borel subgroup of G (see [RS]). The rank of G/H is defined by Panyushev [Pa] and Knop [Kn1]. The minimal rank symmetric spaces were introduced by Brion [Br]. Brion and Joshua have studied the geometry of the closures in X of the B -orbits in G/H , whenever G/H is of minimal rank [BJ]. Tchoudjem has also studied the closures in X of the B -orbits in G/H , whenever G/H is of minimal rank [Tc].

This paper deals with the restriction of equivariant vector bundles on X to some natural class of subvarieties of X , like B -orbit closures.

Let \tilde{G} be the simply connected covering of G . The action of G on X produces an action of \tilde{G} on X using the natural projection $\tilde{G} \rightarrow G$. Given an algebraic vector bundle E on X , we can get a class of vector bundles on X by pulling back E using the automorphisms of X given by the action of G . It can be shown that the isomorphism classes of these pullbacks remain constant if and only if E admits a \tilde{G} -equivariant structure (meaning the action of \tilde{G} on X admits a lift to an action on E).

We prove the following (see Theorem 3.5):

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Theorem 1.1. *Assume that G/H is of minimal rank. Fix a point $x \in G$. Let Z be the closure of BxH/H in X . Let E be a \tilde{G} equivariant vector bundle on X . Then, E is nef (respectively, ample) if and only if the restriction of E to Z is nef (respectively, ample). Similarly, E is trivial if and only if its restriction to Z is trivial.*

In [HMP], a similar result is proved for vector bundles on toric varieties.

Before stating the next result, we recall that for the conjugation action of \tilde{G} on itself, Steinberg proved that for a maximal torus T of G , the restriction homomorphism

$$\mathbb{C}[\tilde{G}]^{\tilde{G}} \longrightarrow \mathbb{C}[\tilde{T}]^{W(G,T)}$$

is an isomorphism, where \tilde{T} is the inverse image of T in \tilde{G} and $W(G, T)$ is the Weyl group of G with respect to T [St1]. Hence, we have the Steinberg map

$$\tau : \tilde{G} \longrightarrow \tilde{T}/W(G, T) = \mathbb{A}^n.$$

Let c be a Coxeter element in the Weyl group $W(G, T)$, and let F be the fiber of the Steinberg map τ containing a representative n_c of c in $N_{\tilde{G}}(\tilde{T})$. Let F' be the image of F in G . Set $\mathcal{Z} = Z_1 \cup Z_2$, where Z_1 is the closure of F' in the wonderful compactification \overline{G} of G , and Z_2 is the unique closed $G \times G$ orbit in \overline{G} [DP].

The group $\tilde{G} \times \tilde{G}$ acts on \overline{G} which factors through the action of $G \times G$ on \overline{G} . Given an algebraic vector bundle E on \overline{G} , the isomorphism classes of its translates by the elements of $G \times G$ remain constant if and only if E admits a $\tilde{G} \times \tilde{G}$ -equivariant structure.

We also prove the following (see Theorem 4.2):

Theorem 1.2. *Let E be a $\tilde{G} \times \tilde{G}$ equivariant vector bundle on \overline{G} . Then, E is nef (respectively, ample) if and only if the restriction of E to \mathcal{Z} is nef (respectively, ample). Similarly, E is trivial if and only if $E|_{\mathcal{Z}}$ is trivial.*

2. PRELIMINARIES

2.1. Lie algebras and Algebraic groups. In this subsection we recall some basic facts and notation on Lie algebras and algebraic groups (see [Hu],[Hu1] for details). Throughout G denotes a semisimple adjoint-type algebraic group over the field \mathbb{C} of complex numbers. In particular, the center of G is trivial. For a maximal torus T of G , the group of all characters of T will be denoted by $X(T)$. The normalizer of T in G will be denoted by $N_G(T)$, while

$$W(G, T) := N_G(T)/T$$

is the Weyl group of G with respect to T . Let $R \subset X(T)$ be the root system of G with respect to T . For a Borel subgroup B of G containing T , let $R^+(B)$ denote the set of positive roots determined by T and B . Further,

$$S = \{\alpha_1, \dots, \alpha_n\}$$

denotes the set of simple roots in $R^+(B)$. For $\alpha \in R^+(B)$, let $s_\alpha \in W(G, T)$ be the reflection corresponding to α . The Lie algebras of G , T and B will be denoted by \mathfrak{g} , \mathfrak{t} and \mathfrak{b} respectively. The dual of the real form $\mathfrak{t}_{\mathbb{R}}$ of \mathfrak{t} is $X(T) \otimes \mathbb{R} = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$.

The positive definite $W(G, T)$ -invariant form on $\text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form on \mathfrak{g} is denoted by $(-, -)$. We use the notation

$$\langle \nu, \alpha \rangle := \frac{2(\nu, \alpha)}{(\alpha, \alpha)}.$$

In this setting one has the Chevalley basis

$$\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in S\}$$

of \mathfrak{g} determined by T . For a root α , we denote by U_α (respectively, \mathfrak{g}_α) the one-dimensional T stable root subgroup of G (respectively, the subspace of \mathfrak{g}) on which T acts through the character α .

Let σ be an algebraic automorphism of G of order two. Let $H = G^\sigma$ be the subgroup consisting of all fixed points of σ in G . The connected component of H containing the identity element will be denoted by H^0 . We refer to [Ri] and [RS] for following facts.

A torus T' of G is said to be σ -anisotropic if $\sigma(t) = t^{-1}$ for every $t \in T'$. Recall that the rank of G/H is defined to be the dimension of a maximal dimensional anisotropic torus.

If T is a σ -stable maximal torus of G , then σ induces an automorphism of $X(T)$ of order two. Note that we have $\sigma(R) = R$. Further, one has $T = T_1 T_2$, where T_1 is a torus such that $\sigma(t) = t$ for every $t \in T_1$, and T_2 is a σ -anisotropic torus. Clearly $T_1 \cap T_2$ is finite. Hence, we have $\text{rank}(G/H) \geq \text{rank}(G) - \text{rank}(H)$.

Throughout, we assume that G/H is of minimal rank, or in other words

$$\text{rank}(G/H) = \text{rank}(G) - \text{rank}(H).$$

We refer to [Br] and [Kn2] for facts about minimal rank.

The following lemma may be known, but for the sake of completeness we provide a proof here.

Lemma 2.2.

- (1) *Any two σ stable maximal tori of G are conjugate by an element of the connected component H^0 of H containing the identity element.*
- (2) *Any maximal torus S of H^0 is contained in a unique maximal torus T of G . Further, this T is σ stable.*
- (3) *Any Borel subgroup Q of H^0 is contained in a σ stable Borel subgroup B of G . Further, the Borel subgroup of G containing Q is unique.*

Proof. Proof of (1). Let T_1 and T_2 be two σ stable maximal tori in G . Define

$$S_i := (T_i \cap H)^0,$$

$i = 1, 2$. Since G/H is of minimal rank, S_1 and S_2 are maximal tori in H^0 . Hence, there is an element $h \in H^0$ such that $hS_1h^{-1} = S_2$. Also, $T_i = C_G(S_i)$ (see [Ri, p. 295, Lemma 5.3 and Lemma 5.4]).

Proof of (2). Take $T = C_G(S)$.

Proof of (3). We will first prove the existence of a stable Borel subgroup containing Q .

By [St2, p. 51, Lemma 7.5], there is a σ stable Borel subgroup B' of G . By [Ri, p. 295, Lemma 5.1], the intersection $(B' \cap H)^0$ is a Borel subgroup of H^0 . Hence, there is a $h \in H^0$ such that $Q = h(B' \cap H)^0 h^{-1}$. Now take $B = hB'h^{-1}$.

To prove the uniqueness of B , let B_1 be a Borel subgroup of G containing Q . As shown above, there is a σ stable Borel subgroup B of G containing Q . Choose a maximal torus S of H^0 lying in Q . From part (2) of the lemma we know that $T = C_G(S)$ is the unique maximal torus of G containing S . Hence, T is contained in both B_1 and B . Thus, there is a $w \in W(G, T)$ such that $wBw^{-1} = B_1$.

We now prove that $R^+(B_1) = R^+(B)$. Let $\alpha \in R^+(B) \setminus R^+(B)^\sigma$. Then the σ invariant vector $x_\alpha + \sigma(x_\alpha)$ is in the Lie algebra of Q . Hence, $x_\alpha + \sigma(x_\alpha)$ is in the Lie algebra of B_1 . Thus, both α and $\sigma(\alpha)$ are in $R^+(B_1) \setminus R^+(B_1)^\sigma$. Hence, we have

$$R^+(B) \setminus R^+(B)^\sigma = R^+(B_1) \setminus R^+(B_1)^\sigma.$$

Now, let $\alpha \in R^+(B)^\sigma$. We will show that σ acts trivially on U_α . Let $T_\alpha \subset T$ be the connected component, containing the identity element, of the kernel of α . Consider the restriction of σ to $C_G(T_\alpha)$. Let C' be the commutator subgroup of $C_G(T_\alpha)$. If the action of σ on U_α is not trivial, it follows that there is a one-dimensional σ stable anisotropic torus S' in C' . Let $T_1 = S'T_\alpha$. Then we have $T_1^\sigma = T_\alpha^\sigma$. Hence by [Ri, Lemma 5.4] we have $T_1 = C_G(T_\alpha^\sigma)$. But this contradicts the fact that T_α^σ is a singular torus. Hence we have $U_\alpha \subset (B)^\sigma = Q \subset B_1$.

Thus, we have shown that $R^+(B) = R^+(B_1)$. Hence, we have $B = B_1$. This completes proof. \square

2.3. Nef vector bundle. Let E be an algebraic vector bundle over a complex projective variety Y . Let $\mathbb{P}(E)$ denote the associated projective bundle over Y whose fiber over any point $y \in Y$ is the space of all one-dimensional quotients of the fiber E_y of E over y . The line bundle over $\mathbb{P}(E)$ whose fiber over any one-dimensional quotient is the one-dimensional quotient itself, will be denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$.

A line bundle L over Y is called *nef* if for every pair (C, φ) , where C is an irreducible smooth complex projective curve and $\varphi : C \rightarrow Y$ is a morphism, the degree of the pullback φ^*L is nonnegative. A vector bundle $E \rightarrow Y$ is called *nef* if the above line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is nef.

3. RESTRICTION OF EQUIVARIANT VECTOR BUNDLES TO B -ORBIT CLOSURE

Let T be a σ stable maximal torus of G . Let B be a Borel subgroup of G containing T such that for any root $\alpha \in R^+(B)$, either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) \in -R^+(B)$.

Let

$$X := \overline{G/H}$$

be the wonderful compactification of the symmetric space G/H constructed in [DP]. Let Z be the closure in X of the B -orbit of a point in G/H .

Let P be the parabolic subgroup of G containing B such that the G/P is the unique closed G orbit in X (see [DP]). In this case, $\sigma(P)$ is opposite to P and $P \cap \sigma(P)$ is the Levi subgroup L of P . Let $R(L)$ denote the roots of L with respect to T .

The following lemma is about a B -orbit in G/H . We refer to [RS] for information on B -orbit closures in G/H . For any algebraic group acting on variety, it is well known that there is always a closed orbit. For instance, any orbit of minimal dimension is closed (see, [Hu1, p. 60. Proposition]).

Lemma 3.1. *Let $x \in G$ be such that $B \cdot xH/H$ is closed in G/H . Then, $x^{-1}Bx$ is σ stable and there is a $w \in W(G, T)$ such that $B \cdot xH/H = B \cdot wH/H$.*

Proof. Let $Q := (x^{-1}Bx \cap H)^0$. Since $B \cdot xH/H$ is closed in G/H , this Q is a Borel subgroup of H^0 . Further, we have $Q \subset x^{-1}Bx$. Hence, by Lemma 2.2, $x^{-1}Bx$ is σ stable.

Now, let $S = (T \cap H)^0$. Since G/H is of minimal rank, this S is a maximal torus in H^0 , and hence we choose a Borel subgroup Q' of H^0 containing S . Thus, there is a $h \in H^0$ such that $hQh^{-1} = Q'$. Consequently, $hx^{-1}Bxh^{-1}$ is a Borel subgroup of G containing T . Thus, there is a $w \in W(G, T)$ and a $b \in B$ such that $xh^{-1} = bw$, and we have $B \cdot xH/H = B \cdot wH/H$. \square

An interesting fact in case of minimal rank is the following uniqueness of the closed B -orbit (see, [Re, p. 1788, Proposition 2.2]).

Lemma 3.2. *There is a unique closed B -orbit in G/H .*

Proof. Clearly, there is a minimal dimensional B -orbit in G/H and it is closed. For its uniqueness, let Bx_1H/H and let Bx_2H/H be two closed B -orbits in G/H . Then, by Lemma 3.1, there are w_1 and w_2 in W such that $B \cdot x_iH/H = B \cdot w_iH/H$ for $i = 1, 2$.

Let $S = (T \cap H)^0$. Set $B_i := w_i^{-1}Bw_i$, and $Q_i = (B_i \cap H)^0$ for $i = 1, 2$. Both Q_1 and Q_2 are Borel subgroups of H^0 containing S . Therefore, there is a $\phi \in W(H^0, S)$ such that $\phi Q_1 \phi^{-1} = Q_2$. Hence both $\phi B_1 \phi^{-1}$ and B_2 are Borel subgroups of G containing Q_2 . By Lemma 2.2, we have $\phi B_1 \phi^{-1} = B_2$, and hence $w_1 = w_2 \phi$. Thus $Bx_1H/H = Bx_2H/H$. \square

We now recall from [BJ] a result of Brion and Joshua.

Lemma 3.3 ([BJ, p. 482, Lemma 2.1.1]). *Let Y be the closure of TH/H in X , and let z denote the unique B -fixed point in X . Then, every T stable curve in X is one of the following:*

- (1) *There is a positive root $\alpha \in R^+(B) \setminus R^+(L)$ and an element $\phi \in W(G, T)$ such that $\phi(C) = C_\alpha = \overline{U_\alpha s_\alpha z}$. In this case α and $\sigma(\alpha)$ are orthogonal, and $s_\alpha s_{\sigma(\alpha)}$ is in $W(H^0, (T \cap H)^0)$.*
- (2) *There is a restricted root $\gamma = \alpha - \sigma(\alpha)$ and an element $\phi \in W(G, T)$ such that $\phi(C) = C_{z, \gamma}$, where $C_{z, \gamma}$ is the unique T -stable curve containing z and on which T acts through the character γ . Moreover, the curve $C_{z, \gamma}$ lies in Y .*

Lemma 3.4. *Take $x \in G$, and let Z be the closure of BxH/H in X . Then every irreducible T stable curve in X lies in $W(G, T) \cdot Z$.*

Proof. Note that the closure of $B \cdot xH/H$ in G/H contains a closed B orbit. Therefore we assume that $B \cdot xH/H$ is the unique closed B orbit in G/H .

By Lemma 3.1, there is an element $w \in W(G, T)$ such that $B \cdot xH/H = B \cdot wH/H$. Let C be an irreducible T stable curve in X . By Lemma 3.3,

- either there is a positive root $\alpha \in R^+(B) \setminus R(L)$ and a $\phi \in W(G, T)$ such that $\phi(C) = C_\alpha = \overline{U_\alpha s_\alpha z}$,
- or there is a restricted root γ and a $\phi \in W(G, T)$ such that $\phi(C) = C_{z, \gamma}$.

Recall that $Y = \overline{TH/H}$ and $S = (T \cap H)^0$. Now, since $s_\alpha s_{\sigma(\alpha)} \in W(H^0, S)$, and $z \in Y$ (see, Lemma 3.3 (2)), we have

$$s_\alpha s_{\sigma(\alpha)} \cdot z \in Y.$$

Hence, $ws_\alpha s_{\sigma(\alpha)} \cdot z \in w \cdot Y = \overline{TwH/H}$. Since α and $\sigma(\alpha)$ are orthogonal, $s_\alpha s_{\sigma(\alpha)}(\alpha) = -\alpha$. Hence, either $w(\alpha)$ is positive or $ws_\alpha s_{\sigma(\alpha)}(\alpha) = w(-\alpha)$ is positive. Further, $s_\alpha s_{\sigma(\alpha)} \in W(H^0, S)$. Hence $BwH/H = Bs_\alpha s_{\sigma(\alpha)}H/H$.

Now, if $w(\alpha)$ is positive, then $U_{w(\alpha)}ws_\alpha s_{\sigma(\alpha)} \cdot z$ is contained in $\overline{BwH/H}$. Hence,

$$ws_{\sigma(\alpha)}(C_\alpha) = \overline{ws_{\sigma(\alpha)}U_\alpha s_\alpha \cdot z} = \overline{U_{w(\alpha)}ws_\alpha s_{\sigma(\alpha)} \cdot z}$$

is contained in $\overline{BwH/H}$.

If $ws_\alpha s_{\sigma(\alpha)}(\alpha) = w(-\alpha)$ is positive, then $ws_\alpha(C_\alpha) = \overline{U_{w(-\alpha)}w \cdot z}$ is contained in $\overline{BwH/H}$.

Thus, in either case, the curve C_α lies in $W(G, T) \cdot Z$.

Since $C_{z, \gamma} \subset Y$, we have $w(C_{z, \gamma}) \subset \overline{TwH/H}$. Hence, both type of curves in Lemma 3.3 lie in the union of the $W(G, T)$ translates of $\overline{BwH/H} = \overline{BxH/H}$. This completes the proof. \square

Notation: Let G be a semi-simple adjoint group over the field \mathbb{C} of complex numbers as above, and let \tilde{G} be its universal cover. For a maximal torus T in G , we denote its inverse image in \tilde{G} by \tilde{T} .

Note that \tilde{G} acts on X and hence we can consider \tilde{G} equivariant vector bundles on X .

Theorem 3.5. *Fix a point $x \in G$. Let Z be the closure of BxH/H in X , where B is a σ stable Borel subgroup of G . Let E be a \tilde{G} equivariant vector bundle on X . Then, E is nef (respectively, ample) if and only if the restriction of E to Z is nef (respectively, ample). Similarly, E is trivial if and only if its restriction to Z is trivial.*

Proof. Since the restriction of a nef or ample or trivial vector bundle to a subvariety is nef or ample or trivial respectively, we have only to prove the “if” part of the theorem.

First assume that the restriction $E|_Z$ is nef. We need to show that for any irreducible closed curve C in $\mathbb{P}(E)$, the degree of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)|_C$ is nonnegative, where $\mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow \mathbb{P}(E)$ is the line bundle defined in Section 2.3.

Let $Y(\tilde{T})$ denote the group of all one-parameter subgroups of \tilde{T} , where \tilde{T} , as before, is the inverse image in \tilde{G} of a σ stable maximal torus T of G lying in B . Choose a \mathbb{Z} -basis $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $Y(\tilde{T})$.

Let \tilde{C} be an irreducible closed curve in the projective bundle $\mathbb{P}(E)$ over X . If the image of \tilde{C} in X is a point, then the degree of $\mathcal{O}_{\mathbb{P}(E)}(1)$ restricted to \tilde{C} is positive, because $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively ample. Hence we can assume that image of \tilde{C} in X is a curve C . Let \tilde{C}_1 be the flat limit of $\lambda_1(t)\tilde{C}$ as t goes to zero (i.e., the one dimensional cycle corresponding to the limit point in the Hilbert Scheme of $\mathbb{P}(E)$). Then \tilde{C}_1 is a 1-dimensional cycle in $\mathbb{P}(E)$ linearly equivalent to \tilde{C} , and the image C_1 of \tilde{C}_1 in X is invariant under λ_1 . Inductively, define \tilde{C}_i to be the flat limit of $\lambda_i(t)\tilde{C}_{i-1}$ as t tends to zero, where $2 \leq i \leq n$. Then \tilde{C}_i is linearly equivalent to \tilde{C} , and the image C_i of \tilde{C}_i in X is invariant under the action on X of the sub-torus of T generated by the images of $\{\lambda_1, \lambda_2, \dots, \lambda_i\}$.

In particular, \tilde{C}_n is linearly equivalent to \tilde{C} , and every irreducible component of \tilde{C}_n lies in the preimage of the T invariant curve $C_n \subset X$. But C_n can be conjugated to a curve in Z (see Lemma 3.4), hence, by our assumption, $E|_{C_n}$ is nef. Therefore, the degree of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}}$ is nonnegative (recall that $\text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}}) = \text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\tilde{C}_n})$). This proves that E is nef.

Next assume that $E|_Z$ is ample.

For any positive integer n , let $\text{Sym}^n(E)$ denote the n -th symmetric power of the equivariant vector bundle E . To prove that E is ample, we first note that there are only finitely many T stable curves in X , and all of them lie in $W(G, T) \cdot Z$ (see Lemma 3.4). Thus the assumption implies that $\text{Sym}^n(E)|_C$ is ample for any T stable curve C in X and for any $n \geq 1$.

Since line bundles on X are equivariant for the \tilde{G} action on X , the vector bundles $\text{Sym}^n(E) \otimes L$ are all \tilde{G} equivariant vector bundles on X , where L is any line bundle on

X . Fix an ample line bundle L on X , and let n be an integer such that $n > \deg(L|_C)$ for every T invariant curve C in X . Then it follows from the argument in the first part of the proof of the theorem that $\text{Sym}^n(E) \otimes L^{-1}|_Z$ is nef, and hence $\text{Sym}^n(E) \otimes L^{-1}$ is nef. This implies $\text{Sym}^n(E)$ is ample and hence E is ample (see, [Ha, p. 67, Proposition 2.4]).

Finally assume that $E|_Z$ is trivial.

Since $E|_Z$ is trivial, the dual $(E|_Z)^* = E^*|_Z$ is also trivial. Note that a trivial vector bundle is nef. Therefore, from the first part of the theorem we conclude that both E and its dual E^* are nef. Therefore, by [DPS, p. 311, Theorem 1.18] the vector bundle E admits a filtration of holomorphic subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

such that each successive quotient E_i/E_{i-1} , $1 \leq i \leq \ell$, admits a unitary flat connection. This implies that E is semistable and $c_j(E) = 0$ for all $j \geq 1$, where c_j is the rational Chern class. Now, by [Si, p. 40, Corollary 3.10] the vector bundle E admits a flat holomorphic connection.

The variety X is simply connected, because it is unirational (see, [Se, p. 483, Proposition 1]). Therefore, any holomorphic vector bundle on X admitting a flat holomorphic connection is a holomorphically trivial vector bundle. In particular, the vector bundle E is trivial. \square

The proof of first two parts of the above theorem closely follows that of [HMP, p.610, Theorem 2.1].

4. A SPECIAL STEINBERG FIBER

As before, G be a semisimple adjoint group. Let T be a maximal torus of G , $W(G, T)$ the Weyl group of G with respect to T and B a Borel subgroup of G containing T . Let \tilde{G} be the simply connected covering of G , and let \tilde{T} (respectively, \tilde{B}) be the inverse image of T (respectively, B) in \tilde{G} . Let c be a Coxeter element in W . We fix a representative n_c of c in $N_{\tilde{G}}(\tilde{T})$.

Lemma 4.1. *The homomorphism $\phi_c : \tilde{T} \rightarrow \tilde{T}$ given by $\phi_c(t) = tn_c t^{-1} n_c^{-1}$ is surjective.*

Proof. It is enough to prove that the kernel of ϕ_c is finite. We can choose a reduced expression $c = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ for c such that $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the set of simple roots labeled in some ordering. Let $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$. Then, the set $\{\beta_1, \beta_2, \dots, \beta_n\}$ is the set of positive roots which are made negative by c^{-1} .

By [YZ, p. 862, Lemma 2.1], we have $\omega_i - c(\omega_i) = \beta_i$. Now, let t be an element of the kernel of ϕ_c . Then, $\beta_i(t) = 1$ for every $i = 1, 2, \dots, n$. Hence,

$$\text{kernel}(\phi_c) \subset \bigcap_{i=1}^n \text{kernel}(\beta_i).$$

Since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a basis of the root lattice of \tilde{G} with respect to \tilde{T} , the kernel of ϕ_c lies in the center of \tilde{G} . Thus, it is finite. \square

Now, let σ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. Note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points, $T \times T$ is a σ -stable maximal torus of $G \times G$ and $B \times B^-$ is a Borel subgroup having the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

Let \overline{G} denote the wonderful compactification of the group G , where G is identified with the symmetric space $(G \times G)/\Delta(G)$.

Now, consider the action of \tilde{G} on \tilde{G} by conjugation. We note that \tilde{T} is stable under the action of $N_{\tilde{G}}(\tilde{T})$.

It is proved in [St1] that the restriction

$$\mathbb{C}[\tilde{G}]^{\tilde{G}} \longrightarrow \mathbb{C}[\tilde{T}]^{W(G, T)}$$

is an isomorphism, and the latter is a polynomial ring. Hence we have the Steinberg map

$$\tau : \tilde{G} \longrightarrow \tilde{T}/W(G, T) = \mathbb{A}^n.$$

Let F be the fiber of the Steinberg map τ containing a representative n_c of c in $N_{\tilde{G}}(\tilde{T})$. By an abuse of notation, we denote by n_c the image of n_c in $N_G(T)$. Let F' be the image of F in G , and let $\mathcal{Z} = Z_1 \cup Z_2$, where Z_1 is the closure of F' in \overline{G} and Z_2 is the unique closed $G \times G$ orbit in \overline{G} .

Theorem 4.2. *Let E be a $\tilde{G} \times \tilde{G}$ -equivariant vector bundle on \overline{G} . Then, E is nef (respectively, ample) if and only if the restriction of E to \mathcal{Z} is nef (respectively, ample). Similarly, E is trivial if and only if its restriction to \mathcal{Z} is so.*

Proof. Set $W = W(G, T)$. By the proof of Theorem 3.5, it is sufficient to prove that every $T \times T$ stable curve in \overline{G} lies in $(W \times W) \cdot \mathcal{Z}$. It is easy to see that, for every root $\alpha \in R^+(B)$, the $T \times T$ stable curve $\overline{(\{1\} \times U_{-\alpha}) \cdot (1, s_\alpha) \cdot z}$ lies in Z_2 . Similarly, $\overline{(U_\alpha \times \{1\}) \cdot (s_\alpha, 1) \cdot z}$ lies in Z_2 for every $\alpha \in R^+(B)$. Thus, every $T \times T$ stable curve of type 1 in Lemma 3.3 lies in $(W \times W) \cdot \mathcal{Z}$.

On the other hand, by Lemma 4.1, the homomorphism ϕ_c is onto and hence, the closure of $Tn_c = \{tn_ct^{-1} \mid t \in T\}$ in \overline{G} is contained in Z_1 . Therefore every $T \times T$ stable curve of type 2 in Lemma 3.3 as well lies in $(W \times W) \cdot \mathcal{Z}$. This completes the proof of the theorem. \square

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